

M2 - Probability Theory : Mid-Semester Exam

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Time Limit : 2.5 hours.

Max. points : 30.

There are two parts to the question paper - PART A and PART B. Read the instructions for each section carefully.

1 PART A : MULTIPLE-CHOICE QUESTIONS - 10 Points.

Please write only the correct choice(s) (for ex., (a), (b) et al.) in your answer scripts. No explanations are needed. Write PART A answers in a separate page.

Some questions will have multiple correct choices. Answer all questions. Each question carries 2 points. 1 point will be awarded if only some correct choices are chosen and no wrong choices are chosen.

- Let $\mathbb{P}$ be a probability measure on $(\mathbb{R}, \mathcal{B})$. Which of the following must be true ?
 - $\mathbb{P}(\{n, n+1, \dots\}) = 1$ for some $n \in \mathbb{Z}$.
 - $\mathbb{P}(\{q\}) = 0$ for all $q \in \mathbb{Q}$.
 - $\lim_{x \downarrow 0} \mathbb{P}((-x, x]) = 0$.
 - For all $\epsilon > 0$, there exists a compact subset $K \subset \mathbb{R}$ such that $\mathbb{P}(K) \geq 1 - \epsilon$.
- For which of the following conditions, is there a probability measure $\mathbb{P}$ on $([0, 1], 2^{[0,1]})$ satisfying the condition.
 - $\mathbb{P}[0, 1/2] = \mathbb{P}[1/2, 1] = 2/3$.
 - $\mathbb{P}[0, 1/2) = \mathbb{P}(1/2, 1] = 2/3$.
 - $\mathbb{P}[0, 1/2) = \mathbb{P}(1/2, 1] = 0$.
 - $\mathbb{P}[0, 1/2] = \mathbb{P}[1/2, 1] = 1/3$.
- Which of the following functions can be a CDF ? ($x_+ := \max\{x, 0\}$).
 - $F(x) = \min\{x_+, 1\}, x \in \mathbb{R}$.
 - $F(x) = \min\{\sqrt{x_+}, 1\}, x \in \mathbb{R}$.
 - $F(x) = \frac{1}{2} + \frac{1}{\pi} \arctan(x), x \in \mathbb{R}$
 - $F(x) = e^{-e^{-x}}, x \in \mathbb{R}$.
- Let X be a random variable with mean and variance 2. Which of the following statements are true?

- (A) $\mathbb{P}(|X - 2| \geq 2) \leq \frac{1}{4}$.
 (B) $\mathbb{P}(|X - 2| \geq 2) \leq \frac{1}{2}$.
 (C) $\mathbb{P}(X \geq 4) \leq \frac{1}{2}$.
 (D) $\mathbb{P}(X \geq 0) = 1$.

5. Which of the following sequences of random variables $X_n, n \geq 1$ are uniformly integrable ?

- (a) X_n are Binomial(n, p_n) with $\sup_{n \geq 1} np_n < 2026$.
 (b) $X_n = \sum_{i=1}^n Y_i$, where Y_i are Exponential(i) random variables.
 (c) X_n are Binomial($n, 1/2$) random variables.
 (d) $X_n = \sum_{i=1}^n Z_i Y_i$, where Y_i are independent Poisson($1/i$) random variables and Z_i are independent Bernoulli($1/i$) random variables, also independent of Y_i 's.

2 PART B : 20 Points.

Answer any two questions only. All questions carry 10 points.

Give necessary justifications and explanations for all your arguments. If you are citing results from the class, mention it clearly.

- Let Y be any non-negative integer-valued random variable. Show that $\mathbb{E}(Yf(Y)) = \lambda \mathbb{E}(f(Y+1))$ for any bounded function $f : \mathbb{N} \cup \{0\} \rightarrow \mathbb{R}$ iff Y has distribution of a Poisson(λ) random variable.
- Let U be uniform random variable and set $X_n = \sin(2\pi n U), n \geq 1$. Find the CDF of X_n for all $n \geq 1$. Show that X_n 's are uncorrelated (i.e., $\mathbb{E}[X_i X_j] = 0$ for $i \neq j$) but not independent.
- Let $X_1, \dots, X_n, \dots, Y_1, \dots, Y_n, \dots$ be i.i.d. Ber(p) random variables. Let Z_n be the length of the longest common subsequence i.e.,

$$Z_n = \max\{k : X_{i_1} = Y_{j_1}, \dots, X_{i_k} = Y_{j_k}, 1 \leq i_1 \leq \dots \leq i_k \leq n, 1 \leq j_1 \leq \dots \leq j_k \leq n\}.$$

Show that there exists a constant $C > 0$ such that

$$\liminf_{n \rightarrow \infty} n^{-1} \mathbb{E}[Z_n] \geq C, \quad \text{Var}[Z_n] \leq n.$$

Does $n^{-1}(Z_n - \mathbb{E}[Z_n])$ converge in probability or a.s. ? If so what is the limit ?

- Let $X_1, \dots, X_n, \dots$ be identically distributed random variables with Exponential(1) distribution. Let $M_n = \max_{i \leq n} X_i$. Show that

$$\limsup_n \frac{M_n}{\log n} \leq 1, \text{ a.s.} \quad (5)$$

Suppose that X_i 's are also independent, then show that

$$\frac{M_n}{\log n} \xrightarrow{p} 1 \text{ as } n \rightarrow \infty. \quad (5)$$

Probability Theory: Notions and Results

Notation & standard distributions

$\mathbb{N}$ natural numbers; $\mathbb{Q}$ rationals; $\mathbb{R}$ reals; $\mathbb{E}, \mathbb{P}, \text{Var}(\cdot)$ expectation, probability, variance.

Bernoulli(p): $\mathbb{P}(X = 1) = p = 1 - \mathbb{P}(X = 0)$.

Binomial(n, p): $\mathbb{P}(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$, $k = 0, \dots, n$.

Poisson(λ): $\mathbb{P}(X = k) = e^{-\lambda} \lambda^k / k!$, $k \in \mathbb{N}$.

Geometric(p): $\mathbb{P}(X = k) = (1 - p)^{k-1} p$, $k = 1, 2, \dots$ (trials to 1st success).

Exponential(λ): density $f(x) = \lambda e^{-\lambda x}$, $x \geq 0$.

Normal(μ, σ^2): density $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$, $x \in \mathbb{R}$.

Inequalities

Markov: $X \geq 0 \Rightarrow \mathbb{P}(X \geq t) \leq \mathbb{E}X/t$.

Chebyshev: $\mathbb{P}(|X - \mathbb{E}X| \geq t) \leq \text{Var}(X)/t^2$.

Chernoff: For $t > 0, \lambda > 0$: $\mathbb{P}(X \geq t) \leq e^{-\lambda t} \mathbb{E}e^{\lambda X}$;

Jensen: φ convex $\Rightarrow \varphi(\mathbb{E}X) \leq \mathbb{E}\varphi(X)$.

Cauchy-Schwarz: $|\mathbb{E}XY| \leq \sqrt{\mathbb{E}X^2} \sqrt{\mathbb{E}Y^2}$.

Hölder: $1/p + 1/q = 1$, $p, q > 1 \Rightarrow \mathbb{E}|XY| \leq \|X\|_p \|Y\|_q$;

Convergence notions

Almost sure: $X_n \xrightarrow{\text{a.s.}} X$ if $\mathbb{P}(\{\omega : X_n(\omega) \rightarrow X(\omega)\}) = 1$.

In probability: $X_n \xrightarrow{p} X$ if $\forall \varepsilon > 0$, $\mathbb{P}(|X_n - X| > \varepsilon) \rightarrow 0$.

In L^p ($p \geq 1$): $\mathbb{E}|X_n - X|^p \rightarrow 0$.

Implications: a.s. $\Rightarrow$ p.; $L^p \Rightarrow$ p. (Markov); p. $\Rightarrow$ a.s. along a subsequence.

Uniform integrability: $\{X_i\}_{i \in I}$ is U.I. if

$$\sup_{i \in I} \mathbb{E}[|X_i| \mathbf{1}(|X_i| > M)] \rightarrow 0 \text{ as } M \rightarrow \infty.$$

Limit theorems for integrals

MCT: $0 \leq X_n \uparrow X$ a.s. $\Rightarrow \mathbb{E}X_n \uparrow \mathbb{E}X$ (allows $+\infty$).

Fatou: $X_n \geq 0 \Rightarrow \mathbb{E}[\liminf X_n] \leq \liminf \mathbb{E}X_n$.

DCT: $X_n \rightarrow X$ a.s., $|X_n| \leq Y$ a.s. with $\mathbb{E}Y < \infty \Rightarrow \mathbb{E}X_n \rightarrow \mathbb{E}X$ (in fact L^1 convergence).

Vitali: $X_n \xrightarrow{p/\text{a.s.}} X$ and $\{X_n\}$ U.I. $\Leftrightarrow X_n \rightarrow X$ in L^1

Weak Law of Large Numbers

$X_1, X_2, \dots$ i.i.d., $\mathbb{E}X_1 = \mu$. Then

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i \rightarrow \mu \text{ in probability.}$$

Borel-Cantelli Lemmas

$A_n \in \mathcal{F}$, $\limsup_n A_n = \bigcap_n \bigcup_{k \geq n} A_k$ (A_n infinitely often).

BC1: $\sum_n \mathbb{P}(A_n) < \infty \Rightarrow \mathbb{P}(\limsup_n A_n) = 0$. (No independence needed.)

BC2: A_n independent and $\sum_n \mathbb{P}(A_n) = \infty \Rightarrow \mathbb{P}(\limsup_n A_n) = 1$.

Together (i.i.d. case): $\mathbb{P}(A_n \text{ i.o.}) \in \{0, 1\}$ — a 0-1 law.

Bernstein's inequality

Hoeffding's lemma: $X \in [a, b]$, $\mathbb{E}X = 0 \Rightarrow \mathbb{E}e^{\lambda X} \leq e^{\lambda^2(b-a)^2/8}$.

$X_1, \dots, X_n$ independent, mean zero, $|X_i| \leq M$ a.s., $\text{Var}(X_i) = \sigma_i^2$, $S_n = \sum_i X_i$, $v = \sum_i \sigma_i^2$. Then, for $t > 0$,

$$\mathbb{P}(S_n \geq t) \leq \exp\left(-\frac{t^2}{2(v + \frac{1}{3}Mt)}\right).$$

Efron-Stein inequality

$X = (X_1, \dots, X_n)$ independent coordinates, $f : \mathcal{X}^n \rightarrow \mathbb{R}$. Let $X^{(i)}$ be X with X_i replaced by an independent copy X'_i . Then

$$\text{Var}(f(X)) \leq \frac{1}{2} \sum_{i=1}^n \mathbb{E}[(f(X) - f(X^{(i)}))^2].$$

Bounded difference inequality

(McDiarmid.) If f satisfies, for each i ,

$$\sup_{x, x'_i} |f(x) - f(x_1, \dots, x'_i, \dots, x_n)| \leq c_i,$$

and $X_1, \dots, X_n$ independent, then

$$\mathbb{P}(f(X) - \mathbb{E}f(X) \geq t) \leq \exp\left(-\frac{2t^2}{\sum_i c_i^2}\right).$$